L06 Jan 15 Countable

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Recall Base $\mathcal{B}$ of a topology $\mathcal{T}$

* $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$
* $\forall x \in G \in \mathcal{T} \ \exists B \in \mathcal{B} \ x \in B \subset G$

Local base $\mathcal{U}_x$ at $x \in X$

$\forall$ nbhd N of x, $\exists U \in \mathcal{U}_x$, $x \in U \subset N$

Second countable C_{II}

First countable C_I

Separable if $\exists$ countable dense set D

$\forall G \in \mathcal{T}, \ G \cap D \neq \emptyset$

Known.

$C_{II} \Longrightarrow C_I$

$C_{II} \Longrightarrow$ Separable

pick $x_j \in B_j \in \mathcal{B}$, $D = \{x_j : j \in \mathbb{N}\}$

Example of $\mathbb{R}^n$, $\{B(q, \frac{1}{n}) : q \in \mathbb{Q}^n\}$

From this example, apparently, if X is C_I and has a countable dense set D, we may have a countable base.

But, it is not so.

Proposition. Separable & metric $\Rightarrow G_{II}$

We have metric, thus

$\{B(x,\frac{1}{n}) : 1 \leq n \in \mathbb{N}\}$ countable at x.

We also have a countable D, $\overline{D} = X$

Naturally, take

$$\mathcal{B} = \{B(q, \tfrac{1}{n}) : 1 \leq n \in \mathbb{N},\ q \in D\}$$

Qu why is it a base?

Need to prove either one.

(1) $\forall\ G \in \mathcal{T}$, $G = \bigcup \mathcal{A}$ for some $\mathcal{A} \subset \mathcal{B}$

(2) $\forall\ x \in G \in \mathcal{T}$, $\exists\ n \in \mathbb{N}, q \in D$, $x \in B(q, \frac{1}{n}) \subset G$

Take any $x \in G \in \mathcal{T}$, so $x \in B(x, \frac{1}{n}) \subset G$

By $\overline{D} = X$, take $q \in B(x, \frac{1}{2n}) \cap D$

then $x \in B(q, \frac{1}{2n}) \subset B(x, \frac{1}{n}) \subset G$

Hence $\mathcal{B}$ is a countable base

Note. Replace $B(q, \frac{1}{n})$ by $U_{q,n}$ where

$\mathcal{U}_q = \{U_{q,n} : n \in \mathbb{N}\}$ is a countable local base

We do not have Δ-inequality to get

$U_{q,2n} \not\subset U_{x,n} \subset G$

Counter-example C_I & separable $\not\Rightarrow C_{II}$

Lower-limit topology on $\mathbb{R}$, generated by $[a,b)$

Let $\mathcal{B}$ be any base for the topology

its element $\bigcup_{\alpha \in I} [a_\alpha, b_\alpha)$,

Take any $x \in \mathbb{R}$ and its nbhd $[x, x+1)$,

As $\mathcal{B}$ is a base, $\exists\ B_x \in \mathcal{B}$ such that

$$x \in B_x \subset [x, x+1)$$

Observe a_α's of B_x

$$x \in B_x \Rightarrow \inf a_\alpha \leq x$$
$$B_x \subset [x, x+1) \Rightarrow \inf a_\alpha \geq x$$

Thus, $x \longmapsto B_x : \mathbb{R} \longrightarrow \mathcal{B}$ is one-one

i.e., $x \neq y \Rightarrow B_x \neq B_y$

Hence $\mathcal{B}$ must be uncountable.

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Given topological spaces $(X, \mathcal{J}_X)$, $(Y, \mathcal{J}_Y)$

and a mapping $f: X \longrightarrow Y$

Qu. How would you define continuity?

Naturally, modify from known situation

For metric spaces, f is continuous at $x_0 \in X$ if

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that

$$d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon$$

Rewrite into set language

$$x \in B_X(x_0, \delta) \Longrightarrow f(x) \in B_Y(f(x_0), \varepsilon)$$

i.e. $$f\big(B_X(x_0, \delta)\big) \subset B_Y(f(x_0), \varepsilon)$$

Now, without metric, there is no balls, nor ε-δ

Definition $f: X \longrightarrow Y$ is continuous at x_0 if

$\forall$ nbhd V of $f(x_0)$, $\exists$ nbhd U of x_0

such that $f(U) \subset V$

Equivalently,

① $\forall\, V \in \underset{\uparrow \mathcal{B}_Y}{\mathcal{J}_Y}$ with $f(x_0) \in V$, $\exists\, U \in \underset{\uparrow \mathcal{B}_X}{\mathcal{J}_X}$, $x_0 \in U$, $f(U) \subset V$

②

① ⇒ ② Take any $f(x_0) \in V \in \mathcal{B}_Y \subset \mathcal{J}_Y$

By ① $\exists\, \underbrace{W \in \mathcal{J}_X,\ x_0 \in W}$, $f(W) \subset V$

$x_0 \in U \subset W$ for some $U \in \mathcal{B}_X$

$$f(U) \subset f(W) \subset V$$

② ⇒ ① Similar.

Qu. What about continuity on the whole X ?

Obvious method: add $\forall\, x \in X \;\; \forall\, V \in \mathcal{T}_Y$ with $f(x) \in V$

$$\underbrace{\exists\, U \in \mathcal{T}_X ,\ \underbrace{x \in U ,\ f(U) \subset V}_{x \in U \subset f^{-1}(V)}}_{f^{-1}(V) \text{ is a nbhd of } x}$$

$x \in f^{-1}(V)$

Equivalently re-written as

$$\forall\, V \in \mathcal{T}_Y \quad \underbrace{\forall\, x \in f^{-1}(V),\ f^{-1}(V) \text{ is a nbhd of } x}_{f^{-1}(V) \text{ is an open set in } X}$$

Definition. $f : X \longrightarrow Y$ is continuous (everywhere) if $\forall\, V \in \mathcal{T}_Y \quad f^{-1}(V) \in \mathcal{T}_X$

Recall the local version

$f : X \longrightarrow Y$ is continuous at $x_0 \in X$ if $\forall\, V \in \mathcal{T}_Y$ with $f(x_0) \in V$, $\exists\, U \in \mathcal{T}_X$ such that $x_0 \in U$ and $f(U) \subset V$